FORMULATION OF A FINITE ELEMENT FRAME WORK FOR A 1-D BAR ELEMENT IN PREDICTING THE DEFLECTION OF A 2-D TRUSS STRUCTURE.

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Abstract
This paper highlights the formulation of a finite element frame work in predicting the deflection in a simple 2-D truss structure. The results obtained from the developed frame work matched the exact analytical solution for the same structure. This was also verified using a standard commercial software (Abaqus). It showed that the accuracy of the finite element frame work (i.e. the computational prediction) in predicting the exact behavior of the truss structure under applied load, depends on the ability to describe accurately the displacement field for the bar element.

Keywords: Deformation, trial solution, Minimum Total Potential Energy, Weighted Residual Formulation

1.0 INTRODUCTION

Structural analysis is generally carried out to assess the behavior of the overall structure. These structures are either simple (for example, truss) or multidimensional (for example: buildings, bridges, dams, airplanes, ships and so on) requiring analysis in different co-ordinate planes under numerous loading scenarios such as temperature, pressure, gravity, suction aerodynamics and so on. In order to get an accurate behavior of these structures under various loading scenarios, the structures would need to be designed, produced and tested to check for failure, the progression of failure as well as when and how the structures are likely to fail (Schroff, 2020). The results obtained from the tests can then be used to improve on the initial design until optimum results are obtained from the tests. This would ideally require designing scaled prototypes or full-scale version of these structures for testing which would be very expensive, time consuming and in some cases not feasible (Schroff, 2020). An economic alternative would be to carry out computational modelling of these structures under the numerous loading scenarios. One way of achieving this computational modelling would to use finite element method to develop and model the engineering problem (Seshu, 2012; Khennane, 2013).

2.0 FINITE ELEMENT FORMULATION METHODOLOGY

Every engineering problem can be described using mathematical models which are developed on the basis of differential equations (referred to as the governing equation) with supporting initial and boundary condition (Kim et al. 2008; Desai, 2011; Sangeetha et al. 2015; Koutromanos, 2018). This governing differential equations can be solved numerically using the finite element method in three stages; pre-processing, processing and post-processing which entails the iterative procedure as shown in Figure 1 (Khennane, 2013; Schroff, 2020).

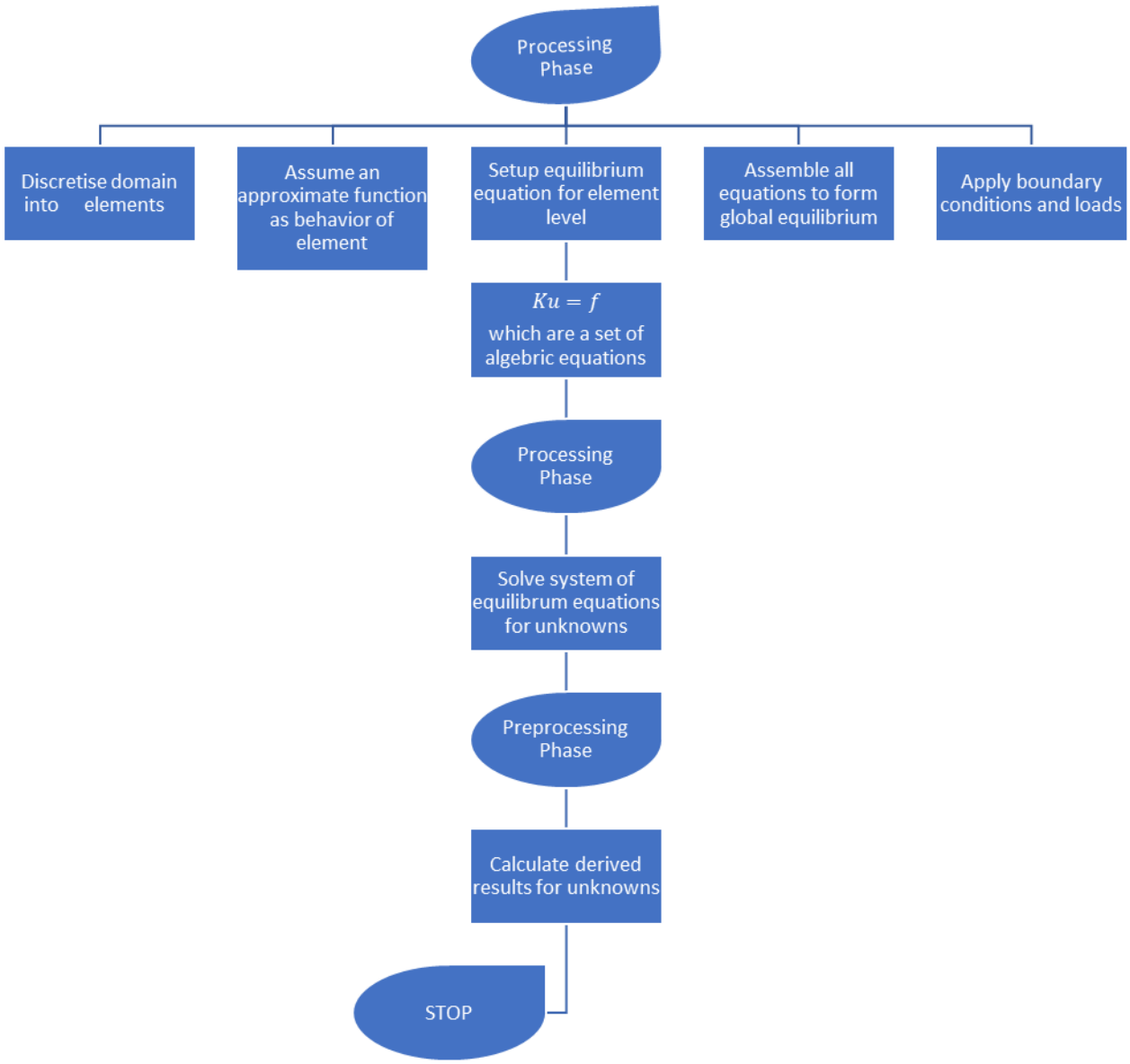


Figure 1: steps involved in finite element problem

3.1 Sectoral (activity-Based) waste temporal distribution

The monthly solid waste generated in each sector is as in Table 3. Figure 7(a-f) show the temporal distribution of MSW in the hostel, academic, administrative, commercial, residential and communal sectors of Unilag respectively. The temporal distribution of MSW revealed that Hostel, Academic and communal sectors exhibited similar trends with two peaks during the year.

3.0 FINITE ELEMENT FORMULATION

A number of techniques are available for formulating finite element problems; the most commonly used techniques are as follows:

1. The Minimum Total Potential Energy Formulation
2. The Weighted Residual Formulation (Ochsner and Merkel, 2005; Kim et al. 2008)

3.1 Minimum Total Potential Energy Formulation

The total potential energy of a system can be described as the variation that exists between an overall strain energy of the system due to internal forces and the work done by external loads in reaching equilibrium. So that for a structural system in stable equilibrium, the minimum total potential energy of the system assumes a stationary value (that is the minimum total potential energy of the system corresponds to a state of displacement in the system when the system is stable) (Seshu, 2012).

Therefore, the stored strain energy in an element is expressed as:

$$U^{(e)} = \int_v \frac{E\varepsilon^2}{2}\, dV \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

While the work done by the external force is expressed as:

$$F^{(e)} = F_i u_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Where:
V —Is the volume of the element
The total potential energy of the system is therefore expressed as:

$$\pi = \sum_e^m U^{(e)} - \sum_i^n F_i u_i \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

With $i = 1,2,\ldots\ldots\ldots\ldots n$

While the minimum total potential is expressed as:

$$\frac{\delta\pi}{\delta u_i} = \frac{\delta}{\delta u_i}\sum_e^m U^{(e)} - \frac{\delta}{\delta u_i}\sum_i^n F_i u_i = 0 \qquad \text{for } i = 1,2,\ldots\ldots n \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

For a single finite element of length l with nodes i and $i+1$ and corresponding displacements of u_i and u_{i+1} the strain will be given as:

$$\varepsilon = \frac{u_{i+1} - u_i}{l} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

In the context of finite element formulation, substituting Equation (5) in Equation (1) results in the strain energy expression for the single finite element given as:

$$U^{(e)} = \int_v \frac{E\varepsilon^2}{2}\, dV = \frac{EA}{2l}\left({u_{i+1}}^2 + {u_i}^2 - 2u_{i+1}u_i\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Minimizing the strain energy with respect to the nodal displacements $u_i$and u_{i+1} as well as the work done by the external actions results in expressions in Equations 6 and 7 respectively:

$$\frac{\delta U^{(e)}}{\delta u_i} = \frac{EA}{l}(u_i - u_{i+1})$$

$$\frac{\delta U^{(e)}}{\delta u_{i+1}} = \frac{EA}{l}(u_{i+1} - u_i) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$\frac{\delta}{\delta u_i} = (F_i u_i + F_{i+1}u_{i+1}) = F_i$$

$$\frac{\delta}{\delta u_{i+1}} = (F_i u_i + F_{i+1} u_{i+1}) = F_{i+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Substituting Equations 6 and 7 in Equation 4 results in

$$k_{eq}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} u_i \\ u_{i+1} \end{pmatrix} - \begin{pmatrix} F_i \\ F_{i+1} \end{pmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.1)$$

The above Equation 8 can be written in a more compact form as:

$$[K]^e(u) = (F) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.2)$$

which is the equilibrium equation for the element (Khennane, 2013; Schroff, 2020).

3.2 Weighted Residual Formulation

This section looks at finite element formulation using weighted residual approach. Physical systems under prescribed loading and boundary conditions can be described using differential equations. The method of weighted residual, requires us assuming approximate solutions for the unknowns in any governing differential equation that describes the physical system for a problem under consideration. For the assumed approximate solution to be valid, it must be continuous and must also satisfy the initial conditions and the boundary conditions for the problem (Kim et al. 2008; Schroff, 2020). Since the assumed solution is only an approximate, the solution to the differential equation is never exact as it likely to have some errors referred to as the residuals. In reducing the residuals to a minimum value, the solution obtained tends to an exact value (Seshu, 2012). Numerous methods (such as the Galrekin, point collocation, least squares and so on) are available for reducing the residuals, but the focus here is on the Galrekin method as it tends to minimize the residuals in an overall sense thoughout the problem domain (Chandrupatla, 2004; Seshu, 2012).

For a 1-D bar element under axial load, the basic steps in the weighted residual approach is given as follows:

a. Setting up the differential equation for the physical system

$$A(x)E\frac{du}{dx} - p = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

With the boundary conditions expressed as:

$$u(0) = 0; \qquad A(x)E\frac{du}{dx}\Big|_{x_i}^{x_j} = P_i \; and P_j$$

b. Assume a suitable approximate trial solution that satisfies the necessary conditions of continuity; initial condition and boundary condition. Hence the assumed solution function for the displacement in Equation 8 is taken as:

$$u(x) = c_1 + c_2 x \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

c. Evaluate the residual by substituting the approximate function into the governing differential equation as given in Equation 9 (Ochsner and Merkel, 2005; Kim t al. 2008; Desai, 2011).

3.3 The Galrekin Method

Minimizing the residual using the Galrekin method to get the solution of the problem under consideration, requires that the residual (R) be orthogonal to the assumed weighting function (W_i). That is:

$$\int_a^b W_i R dx = 0; with\ i = 1,2,3 \dots \dots n \dots \dots \dots \dots \dots \dots \dots (11)$$

Where n is the unknown co-efficient in the approximate solution in Equation 10. Substituting the co-efficients obtained from Equation 11 back into Equation 10 (which is the assumed trial solution), gives the solution to the displacement.

In the context of finite element formulation, (considering the same 1-D dimensional bar element considered in the minimization problem above) we begin by:

1. Establishing the governing differential equation which was already given before in Equation 9 is repeated here as Equation 12:

$$A(x)E\frac{du}{dx} - p = 0 \ \dots \dots \dots \dots \dots \dots \dots \dots (12.1)$$

From Equation 11 the formulation consists of the weighting function (W_i) and the residual (R). The residual (R), is however, obtained by taking the differential of the above governing differential equation that is:

$$\frac{d}{dx}\left(A(x)E\frac{du}{dx}\right) = R \dots \dots \dots \dots \dots \dots \dots \dots (12.2)$$

while the weighting function (W_i) is considered to be the shape function $[N]$ of the particular element being considered. For the above element, the shape function is expressed as:

$$[N] = [N_i \quad N_j] \dots \dots \dots \dots \dots \dots \dots \dots (13.1)$$

Where:

$$N_i = \frac{x_j - x}{l} and \ \ N_j = \frac{x - x_i}{l} \dots \dots \dots \dots \dots \dots \dots \dots (13.2)$$

And the displacement function for the element is given to be:

$$(u)^e = \left[\frac{x_j - x}{l} \qquad N_j = \frac{x - x_i}{l}\right]\begin{pmatrix} u_i \\ u_j \end{pmatrix} \dots \dots \dots \dots \dots \dots \dots \dots (14)$$

Substituting Equations 12.2 and 13.1 back into Equation 11 gives:

$$\int_{x_i}^{x_j} N^T . \frac{d}{dx}\left(AE\frac{du}{dx}\right) dx = 0 \dots \dots \dots \dots \dots \dots \dots \dots (15)$$

Integrating Equation 15 by parts gives:

$$N^T . AE\frac{du}{dx}\Big|_{x_i}^{x_j} - \int_{x_i}^{x_j} \frac{d}{dx}\left(AE\frac{du}{dx}\right).\frac{dN^T}{dx} dx = 0 \dots \dots \dots \dots \dots \dots \dots \dots (16)$$

Evaluating Equation 16 with $x_i = 0$ and $x_j = l$ and the force boundary condition $AE\frac{du}{dx}$ at $x_i$and x_j being equal to $P_i$and P_j respectively, results in the expression given in equation 17

$$\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} u_i \\ u_j \end{pmatrix} = \begin{pmatrix} P_i \\ P_j \end{pmatrix} = 0 \dots \dots \dots \dots \dots \dots \dots \dots (17)$$

This can be further written in a more compact form as given in Equation 17.2 which gives the equilibrium equation for the element,

$$[K]^e(u) = (P) \dots \dots \dots \dots \dots \dots \dots \dots (17.2)$$

(Chandrupatla, 2004; Ochsner and Merkel, 2005; Seshu, 2012; Koutromanos, 2018)

Using the above two methods results in the same equilibrium Equations expressed in 8.2 and 17.2.

4.0 FORMULATION OF THE FINITE ELEMENT FRAME WORK USING PYTHON

For the finite element analysis procedure using Python, the following solution steps were adopted:

1. Calculate the stiffness matrix for each element

$$[K]^e = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (18)$$

2. Build the primary stiffness matrix for the structure
3. Impose the influence of the support conditions
4. Solve for the unknown displacement $\begin{pmatrix} u_{2x} \\ u_{2y} \end{pmatrix}$
5. Substitute back the known displacement into the equilibrium equation expressed in Eqn. 19 to calculate the reaction forces

$$[\boldsymbol{K}](\boldsymbol{U}) = (\boldsymbol{F}) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (19)$$

Where:

$\boldsymbol{K}$ – primary stiffness matrix
$\boldsymbol{U}$ – known displacement
$\boldsymbol{F}$ – force vector

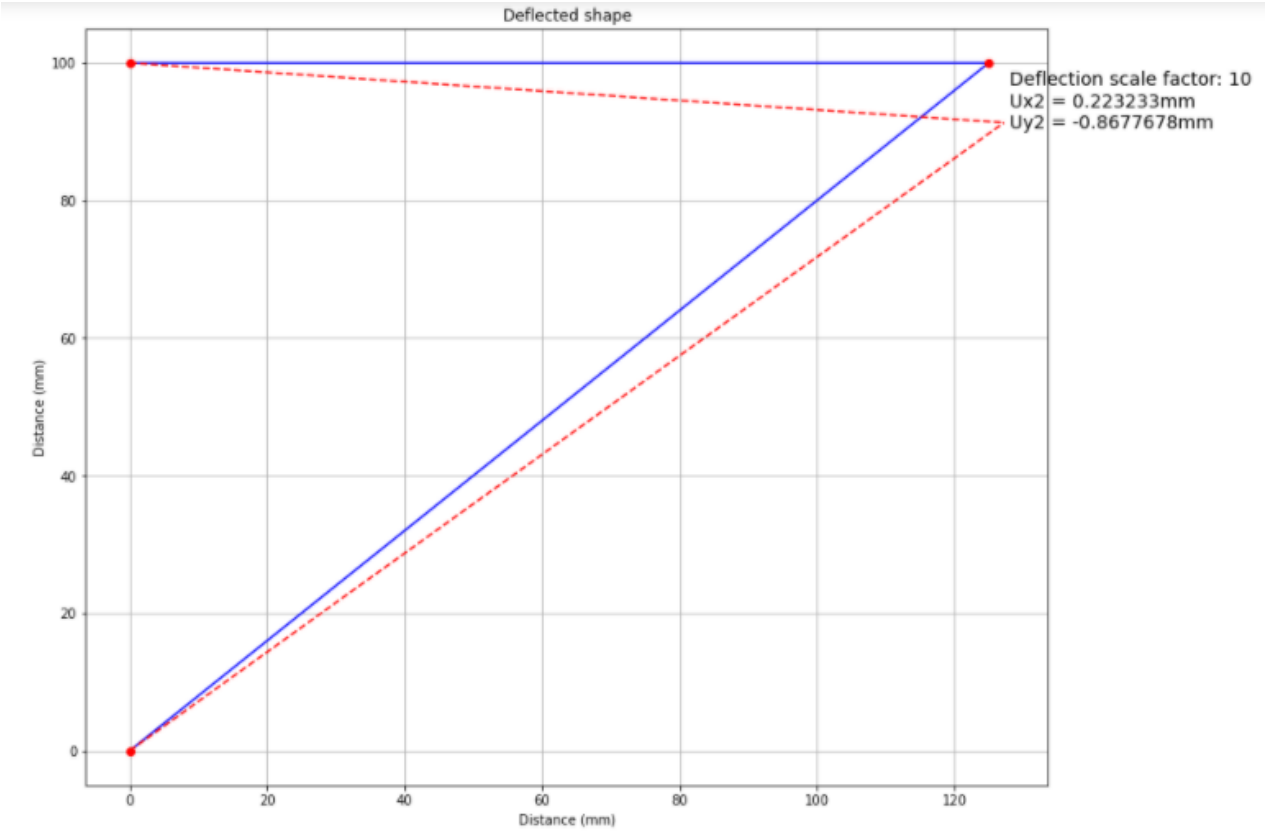


Figure 2: deformed and undeformed shape from the finite element frame work developed using Python program

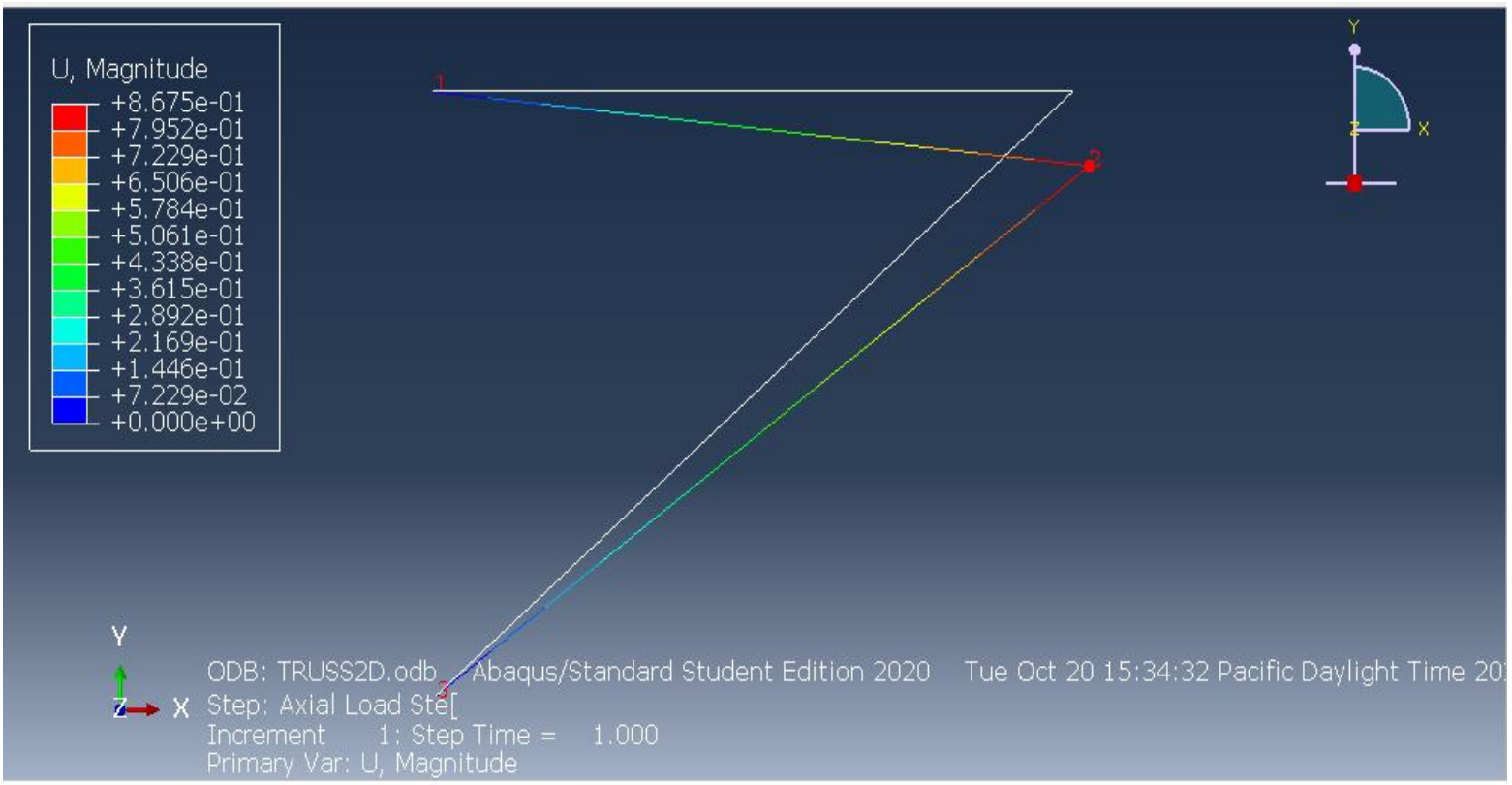


Figure 3: Deformed and undeformed shape simulated using Abaqus

Table 1: Results from an exact analytical solution – material aluminum

Element	Nodes	Length (mm)	Area (mm^2)	Modulus of Elasticity (N/mm^2)	Poisson's Ratio	Nodal Displacements (mm)
A	$(0,1) - (2,3)$	125	10	70000	0.3	$U_{2x} - 0.223$
B	$(2,3) - (4,5)$	160.8	10	70000	0.3	$U_{2y} - 0.867$

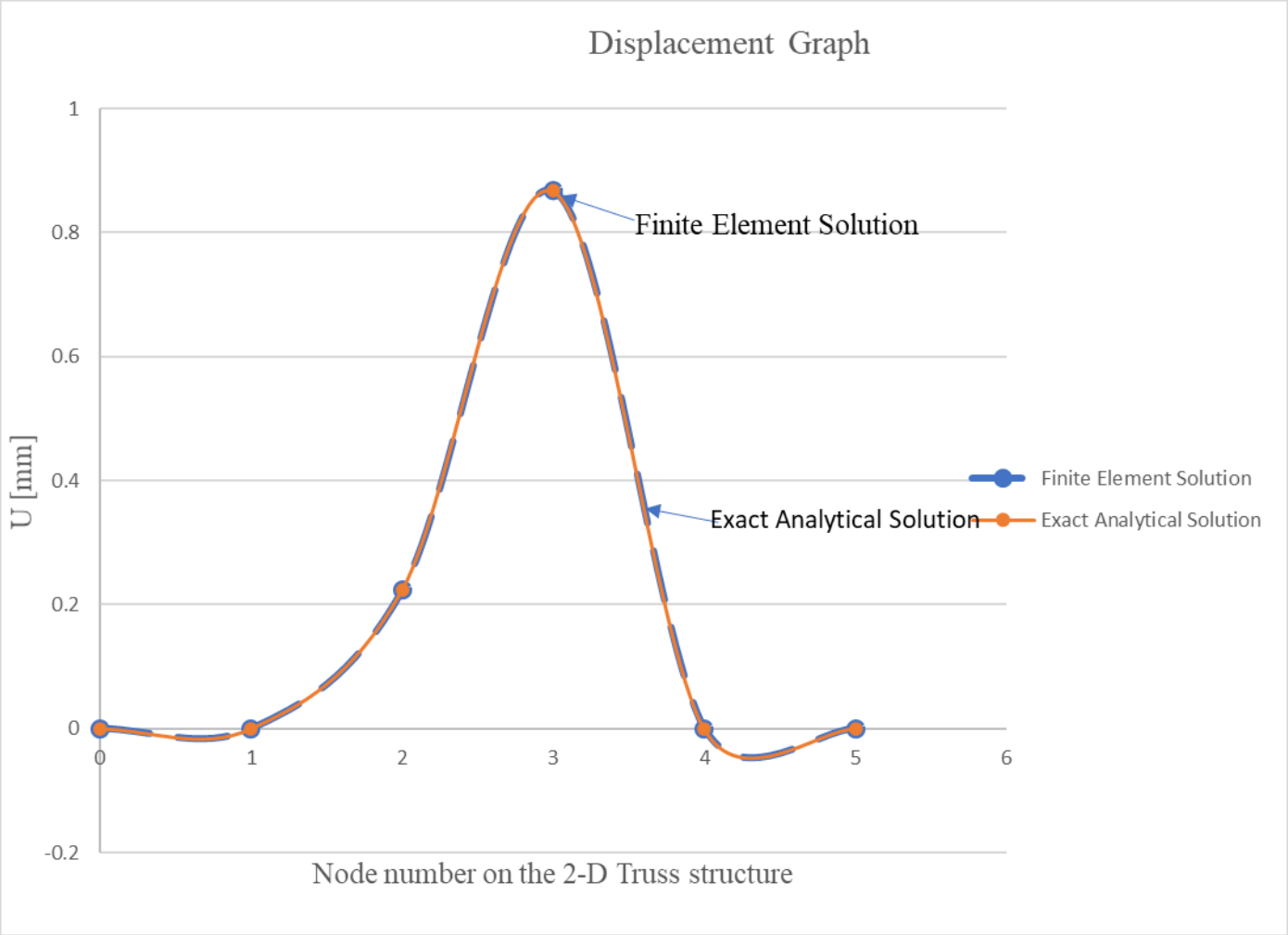


Figure 4: Comparison of exact analytical solution with the finite element formulation

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